CONVERGENCE OF MSE OF A UNIFORM KERNEL ESTIMATOR FOR INTENSITY OF A PERIODIC POISSON PROCESS WITH UNKNOWN PERIOD

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Abstract. Convergence of MSE (Mean-Squared-Error) of a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and proved. The result presented here is a special case of the one in [3]. The aim of this paper is to present an alternative and a relatively simpler proof of convergence for the MSE of the estimator compared to the one in [3]. This is a joint work with R. Helmers and R. Zitikis.

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1. Introduction

In this paper, convergence of MSE (Mean-Squared-Error) of a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and proved. More general results which using general kernel function can be found in [3] and chapter 3 of [4].

Let X be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with unknown period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \qquad (1.1)$$

Suppose that, for some $\omega \in \Omega$, it is only available a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Then, a uniform kernel estimator for λ at a given point $s \in [0, n]$ using only a single realization $X(\omega)$ of the Poisson process X observed in interval $[0, n]$ is presented. (The requirement $s \in [0, n]$ can be dropped if we know the period τ.) Our goals are (a) To determine conditions for having that MSE of this estimator converges to zero, as $n \to \infty$. (b) To present a relatively simpler proof of convergence of the MSE of this estimator compared to the one in [3].

Since λ is a periodic function with period τ, the problem of estimating λ at a given point $s \in [0, n]$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we assume that $s \in [0, \tau)$.

Throughout this paper it is assumed that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s + x) - \lambda(s)| dx = 0$$

(e.g. [7], p.107-108). This assumption is a mild one since the set of all Lebesgue points of λ is dense in $\mathbf{R}$, whenever λ is assumed to be locally integrable.

Let $\hat{\tau}_n$ be any consistent estimator of the period τ, that is,

$$\hat{\tau}_n \xrightarrow{p} \tau,$$

as $n \to \infty$. For example, one may use the estimators constructed in [2] or perhaps the estimator investigated by [6] or [1]. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \tag{1.2}$$

as $n \to \infty$. With these notations, we may define an estimator of $\lambda(s)$ as

$$\hat{\lambda}_n(s) := \frac{\hat{\tau}_n}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n - h_n] \cap [0, n]\right). \tag{1.3}$$

The idea behind the construction of the estimator $\hat{\lambda}_n(s)$ given in (1.3) can be found e.g. in [5].

2. Results

In this section, we focus on convergence of the MSE of $\hat{\lambda}_n$. To obtain our results it is needed an assumption on the estimator $\hat{\tau}_n$ of τ: there exists constant $C > 0$ and positive integer n_0 such that, for all $n \geq n_0$

$$\mathbf{P}\left(\frac{n}{a_n} |\hat{\tau}_n - \tau| \leq C\right) = 1$$

for some fixed sequence $a_n \downarrow 0$. The shorthand notation for this assumption will be :

$$n\,|\hat{\tau}_n - \tau| = \mathcal{O}(a_n)$$

with probability 1, as $n \to \infty$.

The main results of this paper are the following theorems.

Theorem 2.1. *Suppose that λ is periodic and locally integrable. If, in addition, (1.2) holds, and*

$$n\left|\hat{\tau}_n - \tau\right| = \mathcal{O}\left(\delta_n h_n\right) \tag{2.1}$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$, then

$$\mathbf{E}\hat{\lambda}_n(s) \ \to \ \lambda(s) \tag{2.2}$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_n(s)$ is an asymptotically unbiased estimator of $\lambda(s)$.

Note that the requirement $nh_n \to \infty$ as $n \to \infty$, which is needed to obtain weak consistency of $\hat{\lambda}_n$ (cf. [5]), is not needed to establish asymptotic unbiasedness of $\hat{\lambda}_n$, i.e. (2.2).

Theorem 2.2. *Suppose that λ is periodic and locally integrable. If, in addition, (1.2) and (2.1) hold true and*

$$nh_n \to \infty \tag{2.3}$$

as $n \to \infty$, then

$$Var\left(\hat{\lambda}_n(s)\right) = o(1) \tag{2.4}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

By Theorem 2.1 and Theorem 2.2 we obtain the following corollary.

Corollary 2.3. *Suppose that λ is periodic and locally integrable. If conditions (1.2), (2.1) and (2.3) hold true, then*

$$MSE\left(\hat{\lambda}_n(s)\right) = Var\left(\hat{\lambda}_n(s)\right) + Bias^2\left(\hat{\lambda}_n(s)\right) \to 0 \tag{2.5}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

3. Proof of Theorem 2.1

We begin with two simple lemmas, which will be useful in establishing our results.

Lemma 3.1. *Suppose that λ is periodic and locally integrable. If the bandwidth h_n be such that (1.2) holds true, then*

$$\mathbf{E}\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right) \to \lambda(s) \tag{3.1}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: Using the fact that X is Poisson, the l.h.s. of (3.1) can be written as

$$\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in[0,n])dx$$

$$=\frac{\tau}{2nh_n}\int_{-h_n}^{h_n}\lambda(s+x)\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])dx. \qquad (3.2)$$

Now note that

$$\left(\frac{n}{\tau}-1\right)\leq\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])\leq\left(\frac{n}{\tau}+1\right),$$

which implies

$$\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])$$

can be written as $(1+\mathcal{O}(n^{-1}))$, as $n\to\infty$, uniformly in x. Then, the quantity on the r.h.s. of (3.2) can be written as

$$\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+x)dx. \qquad (3.3)$$

By (1.2) together with the assumption that s is a Lebesgue point of λ, we have that

$$(2h_n)^{-1}\int_{-h_n}^{h_n}\lambda(s+x)dx=\lambda(s)+o(1),$$

as $n\to\infty$. Then we obtain this lemma. This completes the proof of Lemma 3.1.

Lemma 3.2. *Suppose that the assumption (2.1) is satisfied. Then, for each positive integer m, we have that*

$$\mathbf{E}\left(\hat{\tau}_n-\tau\right)^{2m}=\mathcal{O}\left(n^{-2m}\delta_n^{2m}h_n^{2m}\right) \qquad (3.4)$$

as $n\to\infty$.

Proof: By the assumption (2.1), there exists large positive constant C and positive integer n_0 such that

$$|\hat{\tau}_n-\tau|\leq Cn^{-1}\delta_n h_n, \qquad (3.5)$$

with probability 1, for all $n\geq n_0$. Then, the l.h.s. of (3.4) can be written as

$$\int_0^{\infty}x^{2m}d\mathbf{P}\left(|\hat{\tau}_n-\tau|\leq x\right)$$

$$=-\int_0^{Cn^{-1}\delta_n h_n}x^{2m}d\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right). \qquad (3.6)$$

By partial integration, the r.h.s. of (3.6) is equal to

$$\begin{aligned} & -x^{2m}\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right)|_0^{Cn^{-1}\delta_n h_n} \\ & +2m\int_0^{Cn^{-1}\delta_n h_n}\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right)x^{2m-1}dx. \end{aligned} \tag{3.7}$$

The first term of (3.7) is equals to zero, while its second term is at most equal to

$$\begin{aligned} 2m\int_0^{Cn^{-1}\delta_n h_n} x^{2m-1}dx &= C^{2m}n^{-2m}\delta_n^{2m}h_n^{2m} \\ &= O\left(n^{-2m}\delta_n^{2m}h_n^{2m}\right), \end{aligned} \tag{3.8}$$

as $n\to\infty$. This completes the proof of Lemma 3.2.

Proof of Theorem 2.1

We will prove (2.2) by showing

$$\mathbf{E}\hat{\lambda}_n(s)=\lambda(s)+o(1), \tag{3.9}$$

as $n\to\infty$. First we write $\mathbf{E}\hat{\lambda}_n(s)$ as

$$\begin{aligned} & \left(\mathbf{E}\hat{\lambda}_n(s)-\mathbf{E}\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)\right) \\ & +\frac{\tau}{2nh_n}\left(\mathbf{E}\sum_{k=-\infty}^{\infty}X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)\right. \\ & \left.-\mathbf{E}\sum_{k=-\infty}^{\infty}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right) \\ & +\mathbf{E}\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right). \end{aligned} \tag{3.10}$$

By Lemma 3.1, we have that the third term of (3.10) is equal to $\lambda(s)+o(1)$, as $n\to\infty$. Hence, to prove (3.9), it remains to check that both the first and second terms of (3.10) are $o(1)$, as $n\to\infty$.

First we consider the first term of (3.10). The absolute value of this term can be written as

$$\begin{aligned} & \frac{1}{2nh_n}\left|\mathbf{E}\left(\hat{\tau}_n-\tau\right)\sum_{k=-\infty}^{\infty}X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)\right| \\ & \leq\frac{1}{2nh_n}\left(\mathbf{E}\left(\hat{\tau}_n-\tau\right)^2\right)^{\frac{1}{2}}\left(\mathbf{E}\left(\sum_{k=-\infty}^{\infty}X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)\right)^2\right)^{\frac{1}{2}}, \end{aligned} \tag{3.11}$$

(by Cauchy-Schwarz inequality). By Lemma 3.2 with $m = 1$ (we take $\delta_n = 1$), we have that the square-root of the first expectation on the r.h.s. of (3.11) is of order $\mathcal{O}(n^{-1}h_n)$, as $n \to \infty$. For large n, by (1.2) and (2.1), the intervals $(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n])$ and $(B_{h_n}(s + j\hat{\tau}_n) \cap [0, n])$ are disjoint with probability 1, provided $k \neq j$. Hence, for large n, we have with probability 1 that

$$\sum_{k=-\infty}^{\infty} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n]\right) \leq X([0, n]). \tag{3.12}$$

Then the square-root of the second expectation on the r.h.s. of (3.11) does not exceed $(\mathbf{E}X^2([0, n]))^{\frac{1}{2}} = \mathcal{O}(n)$, as $n \to \infty$. Hence, we have that the r.h.s. of (3.11) is of order $\mathcal{O}(n^{-1})$, which is $o(1)$, as $n \to \infty$.

Next we consider the second term of (3.10). By Fubini's, the absolute value of this term can be written as

$$\frac{\tau}{2nh_n}\left|\sum_{k=-\infty}^{\infty} \mathbf{E}\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n]\right) - X\left(B_{h_n}(s + k\tau) \cap [0, n]\right)\right\}\right|. \tag{3.13}$$

Now note that the difference within curly brackets of (3.13) does not exceed

$$X\left(B_{h_n}(s + k\hat{\tau}_n)\Delta B_{h_n}(s + k\tau) \cap [0, n]\right). \tag{3.14}$$

We notice that

$$B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \subseteq B_{h_n}(s + k\hat{\tau}_n) \subseteq B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau). \tag{3.15}$$

By (3.14) and (3.15) we have

$$\begin{aligned} &\left|\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n]\right) - X\left(B_{h_n}(s + k\tau) \cap [0, n]\right)\right\}\right| \\ &\leq 2X\left(B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau) \setminus B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \cap [0, n]\right). \end{aligned} \tag{3.16}$$

By (3.16), the quantity in (3.13) does not exceed

$$\frac{\tau}{nh_n}\sum_{k=-\infty}^{\infty} \mathbf{E}X\left(B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau) \setminus B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \cap [0, n]\right). \tag{3.17}$$

Since $s \in [0, n]$, by condition (2.1), we have with probability 1 that the magnitude of any integer k such that $\{s + k\hat{\tau}_n + [-h_n, h_n]\} \cap [0, n] \neq \emptyset$ is at most of order $\mathcal{O}(n)$, as $n \to \infty$. By (2.1), there exists large fixed positive integer n_0 and positive constant C, such that

$$n|\hat{\tau}_n - \tau| \leq C\delta_n h_n \tag{3.18}$$

with probability 1, for all $n \geq n_0$. Then, for large n, the quantity in (3.17) does not exceed

$$\frac{\tau}{nh_n} \sum_{k=-\infty}^{\infty} \mathbf{E}X \left(B_{(1+C\delta_n)h_n}(s+k\tau) \setminus B_{(1-C\delta_n)h_n}(s+k\tau) \cap [0,n] \right)$$
$$= \frac{\tau}{nh_n} \sum_{k=-\infty}^{\infty} \int_{B_{(1+C\delta_n)h_n}(0) \setminus B_{(1-C\delta_n)h_n}(0)} \lambda(x+s+k\tau)$$
$$\mathbf{I}(x+s+k\tau \in [0,n])dx$$
$$\leq \frac{\tau}{h_n} \left(\frac{N_n+1}{n} \right) \int_{B_{(1+C\delta_n)h_n}(0) \setminus B_{(1-C\delta_n)h_n}(0)} \lambda(x+s)dx$$
$$\leq \frac{2}{h_n} \int_{B_{(1+C\delta_n)h_n}(0)} |\lambda(s+x) - \lambda(s)|dx$$
$$+ \frac{2\lambda(s)}{h_n} |B_{(1+C\delta_n)h_n}(0) \setminus B_{(1-C\delta_n)h_n}(0)|. \tag{3.19}$$

To get the upper bound on the r.h.s. of (3.19), we use the fact that $\lambda(x+s+k\tau) = \lambda(x+s)$ (by periodicity of λ), $\sum_{k=-\infty}^{\infty} \mathbf{I}(x+s+k\tau \in [0,n]) \leq N_n + 1$, and for large n, we also have that $(N_n+1)/n \leq \frac{2}{\tau}$. Since s is a Lebesgue point of λ, the first term on the r.h.s. of (3.19) is $o(1)$, as $n \to \infty$. While its second term does not exceed $8C\delta_n\lambda(s)$, which is also $o(1)$, as $n \to \infty$. Then we have that the second term of (3.10) is $o(1)$, as $n \to \infty$. This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

First we write

$$Var\left(\hat{\lambda}_n(s)\right) = \mathbf{E}\left(\hat{\lambda}_n(s)\right)^2 - \left(\mathbf{E}\hat{\lambda}_n(s)\right)^2. \tag{4.1}$$

Since by Theorem 2.1, the second term on the r.h.s. of (4.1) is equal to $-\lambda^2(s)+o(1)$ as $n \to \infty$, to prove this theorem, it suffices to show that the first term on the r.h.s. of (4.1) is equal to $\lambda^2(s)+o(1)$ as $n \to \infty$.

The first term on the r.h.s. of (4.1) can be written as

$$\frac{1}{n^2h_n^2}\mathbf{E}\left(\hat{\tau}_n - \tau\right)^2 \left(\sum_{k=-\infty}^{\infty} X\left(B_{h_n}(s+k\hat{\tau}_n) \cap [0,n]\right) \right)^2$$
$$+ \frac{2\tau}{n^2h_n^2}\mathbf{E}\left(\hat{\tau}_n - \tau\right) \left(\sum_{k=-\infty}^{\infty} X\left(B_{h_n}(s+k\hat{\tau}_n) \cap [0,n]\right) \right)^2$$
$$+ \frac{\tau^2}{n^2h_n^2}\mathbf{E} \left(\sum_{k=-\infty}^{\infty} X\left(B_{h_n}(s+k\hat{\tau}_n) \cap [0,n]\right) \right)^2. \tag{4.2}$$

We will show that the third term of (4.2) is equal to $\lambda^2(s)+o(1)$, while the other terms are $o(1)$ as $n \to \infty$.

First we consider the first term of (4.2). By (3.12) and Cauchy-Schwarz inequality, this term does not exceed

$$\left(n^2h_n^2\right)^{-1}\left(\mathbf{E}(\hat{\tau}_n-\tau)^4\right)^{\frac{1}{2}}\left(\mathbf{E}X^4([0,n])\right)^{\frac{1}{2}}.$$

We know that $(\mathbf{E}X^4([0,n]))^{\frac{1}{2}} = \mathcal{O}(n^2)$ as $n\to\infty$. By Lemma 3.2 for $m=2$ (we take $\delta_n=1$), we have that $(\mathbf{E}(\hat{\tau}_n-\tau)^4)^{\frac{1}{2}} = \mathcal{O}(n^{-2}h_n^2)$ as $n\to\infty$. Hence, the first term of (4.2) is of order $\mathcal{O}(n^{-2})$, which is $o(1)$ as $n\to\infty$. Using a similar argument, by noting now that $(\mathbf{E}(\hat{\tau}_n-\tau)^2)^{\frac{1}{2}} = \mathcal{O}(n^{-1}\delta_n h_n)$ as $n\to\infty$, we have the second term of (4.2) is of order $o(n^{-1}h_n^{-1})$, which (by assumption (2.3)) is $o(1)$ as $n\to\infty$.

Next we consider the third term of (4.2). This term can be written as

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2}\ \mathbf{E}\ \left(\sum_{k=-\infty}^{\infty}\left[X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)-X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right]\right)^2\\
&+\frac{2\tau^2}{n^2h_n^2}\ \mathbf{E}\ \left(\sum_{k=-\infty}^{\infty}\left[X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)-X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right]\right.\\
&\qquad\qquad\left.\sum_{l=-\infty}^{\infty}X\left(B_{h_n}(s+l\tau)\cap[0,n]\right)\right)\\
&+\frac{\tau^2}{n^2h_n^2}\ \mathbf{E}\ \left(\sum_{k=-\infty}^{\infty}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right)^2 \qquad (4.3)
\end{aligned}$$

We will show that the third term of (4.3) is equal to $\lambda^2(s)+o(1)$, while the other terms are $o(1)$ as $n\to\infty$.

First we consider the first term of (4.3). By (3.16) and (3.18), the expectation appearing in this term does not exceed

$$\mathbf{E}\left(\sum_{k=-\infty}^{\infty}2X\left(B_{(1+C\delta_n)h_n}(s+k\tau)\setminus B_{(1-C\delta_n)h_n}(s+k\tau)\cap[0,n]\right)\right)^2. \quad (4.4)$$

By writing square of a sum as a double sum, we can interchange summations and expectation. Then we distinguish two cases, namely the case where the indexes are the same and the case where the indexes are different. For sufficiently large n, since $h_n\downarrow 0$ and $\delta_n\downarrow 0$ as $n\to\infty$,

$$2X\left(B_{(1+C\delta_n)h_n}(s+k\tau)\setminus B_{(1-C\delta_n)h_n}(s+k\tau)\cap[0,n]\right)$$

and

$$2X\left(B_{(1+C\delta_n)h_n}(s+j\tau)\setminus B_{(1-C\delta_n)h_n}(s+j\tau)\cap[0,n]\right)$$

are independent, provided $k \neq j$. Then, for large n, the expectation in (4.4) does not exceed

$$
\begin{aligned}
&4 \sum_{k=-\infty}^{\infty} \mathbf{E}X^2 \left(B_{(1+C\delta_n)h_n}(s+k\tau) \setminus B_{(1-C\delta_n)h_n}(s+k\tau) \cap [0,n]\right) \\
&+4 \left(\sum_{k=-\infty}^{\infty} \mathbf{E}X \left(B_{(1+C\delta_n)h_n}(s+k\tau) \setminus B_{(1-C\delta_n)h_n}(s+k\tau) \cap [0,n]\right) \right)^2 \\
&\leq 8 \left(\sum_{k=-\infty}^{\infty} \mathbf{E}X \left(B_{(1+C\delta_n)h_n}(s+k\tau) \setminus B_{(1-C\delta_n)h_n}(s+k\tau) \cap [0,n]\right) \right)^2 \\
&\leq 8 \left(N_n+1\right)^2 \left(\int_{B_{(1+C\delta_n)h_n}(0) \setminus B_{(1-C\delta_n)h_n}(0)} \lambda(x+s)dx \right)^2, \qquad (4.5)
\end{aligned}
$$

by a similar argument as the one used in (3.19). Hence, to show that the first term of (4.3) is $o(1)$ as $n \to \infty$, it suffices now to check

$$
\tau^2 \left(\frac{N_n+1}{n}\right)^2 \left(\frac{1}{h_n} \int_{B_{(1+C\delta_n)h_n}(0) \setminus B_{(1-C\delta_n)h_n}(0)} \lambda(x+s)dx \right)^2 = o(1), \quad (4.6)
$$

as $n \to \infty$. But, by a similar argument as the one used in (3.19) and the paragraph following it, it is clear that we have (4.6).

A similar argument, together with an application of Cauchy-Schwarz inequality, shows that the second term of (4.3) is $o(1)$ as $n \to \infty$.

It remains to show that the third term of (4.3) is equal to $\lambda^2(s)+o(1)$, as $n \to \infty$. To do this we argue as follows. By writing square of a sum as double sums, we can interchange summations and expectation. Recall that for large n, the intervals $(B_{h_n}(s+k\tau) \cap [0,n])$ and $(B_{h_n}(s+j\tau) \cap [0,n])$, for $k \neq j$, are disjoint. Then, the expectation appearing in this term can be written as

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \mathbf{E}X \left(B_{h_n}(s+k\tau) \cap [0,n]\right) X \left(B_{h_n}(s+j\tau) \cap [0,n]\right) \\
&= \sum_{k=-\infty}^{\infty} \mathbf{E}X^2 \left(B_{h_n}(s+k\tau) \cap [0,n]\right) \\
&+ \sum_{k \neq j}^{\infty} \sum^{\infty} \left(\mathbf{E}X \left(B_{h_n}(s+k\tau) \cap [0,n]\right)\right) \left(\mathbf{E}X \left(B_{h_n}(s+j\tau) \cap [0,n]\right)\right) \\
&= \sum_{k=-\infty}^{\infty} \mathbf{E}X \left(B_{h_n}(s+k\tau) \cap [0,n]\right) \\
&+ \left(\sum_{k=-\infty}^{\infty} \mathbf{E}X \left(B_{h_n}(s+k\tau) \cap [0,n]\right) \right)^2,
\end{aligned}
$$

(4.7)

because X is Poisson so that $\mathbf{E}X^2(.) = \mathbf{E}X(.) + (\mathbf{E}X(.))^2$. By (4.7), the quantity in the third term of (4.3) can be written as

$$\left(\frac{\tau}{2nh_n}\right)\left(\mathbf{E}\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right)$$
$$+\left(\mathbf{E}\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right)^2. \tag{4.8}$$

By Lemma 3.1 and assumption (2.3), we have that the first term of (4.8) is $o(1)$, as $n\to\infty$. Lemma 3.1 also shows that the second term of (4.8) is equal to $\lambda^2(s)+o(1)$, as $n\to\infty$. This completes the proof Theorem 2.2.

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